Синхронизация при модуляции FM 16

$\overline{\Lambda(\psi)} = \int_{-x}^{x} \Lambda(\psi)p(A,B)dAdB$ — усредненная функция правдобия в общем виде, где $\Lambda(\psi) = \exp\left[\frac{2}{N_0}\int_0^T r(t)s(t,\varphi|\psi)\right]dt$, а ψ – искомая оценка фазы.
$s(t,\varphi) = \cos(2\pi f_c t + \psi + \varphi_k)$. $A_k = \cos\varphi_k$, $B = \sin\varphi_k$.

Полярные координаты связаны с декартовыми соотношениями: $x = r\cos\varphi$, $y = r\sin\varphi$.

Якобиан по определению $\frac{D(x,y)}{D(\xi,\eta)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix}$, тогда $\frac{D(x,y)}{D(r,\varphi)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$, обратно, якобиан будет равен $\frac{1}{r}$

В общем случае переход к системе координат в двойном интеграле: $\iint_G dxdy = \iint_\Gamma |J|\, drd\varphi = \iint_\Gamma r dr d\varphi$, где Γ — область изменения r, φ.

Рассмотрим случай ФМ 16.

В случае полярной системы координат плотность вероятности: $p(r,\varphi) = \frac{1}{16}\sum\limits_{i=1}^{16}\delta(\varphi-\varphi_i)\cdot r\delta(r-1)$. Где $\varphi_i = 0, \frac{\pi}{8} \ldots \frac{11\pi}{8}$ – 16 значений фазы, соответсвующих $M = 16$.
Тогда $\overline{\Lambda(\psi)} = \frac{1}{16}\int_{-x}^{x}\Lambda(\psi)\delta(\varphi-\varphi_i)\delta(r-1)rdrd\varphi$
Поскольку $r = 1$, и $s(t,\varphi,\psi)$ от r не зависит, то интеграл переписывается в виде:
$\frac{1}{16}\int_{-x}^{x}\Lambda(\psi)\sum\delta(\varphi-\varphi_i)d\varphi$.

$$\overline{\Lambda(\psi)} = \frac{1}{16}\sum_{i=1}^{16}\exp\left[\frac{2}{N_0}\int_0^T r(t)s(t,\varphi_i|\psi)dt\right] = \frac{1}{16}\cdot\left[\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\right) + \right.$$
$$2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{8})dt\right) + 2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{4})dt\right) +$$
$$2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{3\pi}{8})dt\right) + \exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{2})dt\right) +$$
$$2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{5\pi}{8})dt\right) + 2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{3\pi}{4})dt\right) +$$
$$2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{7\pi}{8})dt\right) + \exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\pi)dt\right) +$$
$$\left.\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{3\pi}{2})dt\right)\right]$$

По формулам приведения $\cos(\varphi) = -\cos(\varphi+\pi)$, $\cos(2\pi-\varphi) = \cos(\varphi)$. Также, $\frac{1}{2}\exp\left(\cos(\varphi)+\cos(\varphi+\pi)\right) = \mathrm{ch}(\varphi)$.

$$\overline{\Lambda(\psi)} = \frac{1}{16}\cdot\left[\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\right) + \exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\pi)dt\right) + \right.$$
$$2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{8})dt\right) + 2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{7\pi}{8})dt\right) +$$
$$2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{4})dt\right) + 2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{3\pi}{4})dt\right) +$$
$$2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{3\pi}{8})dt\right) + 2\cdot\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{5\pi}{8})dt\right) +$$
$$\left.\exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{2})dt\right) + \exp\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{3\pi}{2})dt\right)\right]$$

Сгруппировав данным образом, получим

$$\overline{\Lambda(\psi)} = \frac{1}{16}\cdot\left[\mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\right) + \right.$$
$$2\,\mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{8})dt\right) + 2\,\mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{4})dt\right) +$$
$$\left.2\,\mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{3\pi}{8})dt\right) + \mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{2})dt\right)\right].$$

Тогда логарифм функции правдоподобия:

$$\ln\left\{\frac{1}{16}\cdot\left[\mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\right) + 2\,\mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{8}))dt\right) + \right.\right.$$
$$2\,\mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{\pi}{4}))dt\right) + 2\,\mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\tfrac{3\pi}{8}))dt\right) +$$
$$\left.\left.\mathrm{ch}\left(\frac{2}{N_0}\int_0^T r(t)\sin(2\pi f_c t+\psi)dt\right)\right]\right\}.$$

Теперь необходимо усреднить функцию правдоподобия по k реализациям:

$\overline{\Lambda}(\psi) = C\exp\left\{\sum\limits_{n=0}^{k-1}\frac{1}{16}\cdot\left[\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\right)+\right.\right.$
$2\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\frac{\pi}{8})dt\right)+2\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\frac{\pi}{4})dt\right)+$
$\left.\left.2\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\frac{3\pi}{8})dt\right)+\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\right)\right]\right\}.$

Возьмем теперь логарифм ФП, продифференцируем его по ψ и приравняем результат нулю.

$\ln\overline{\Lambda}(\psi)=\sum\limits_{n=0}^{k-1}\frac{1}{16}\cdot\left[\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\right)+2\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\frac{\pi}{8})dt\right)+\right.$
$2\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\frac{\pi}{4})dt\right)+2\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\frac{3\pi}{8})dt\right)+$
$\left.\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\sin(2\pi f_c t+\psi)dt\right)\right].$

Далее предлагается три подхода по оценке, основанные на аппроксимации вида:

1) способ.

Запишем $\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\right)$ как

$$\operatorname{ch}\left(\frac{2}{N_0}\cos\psi\int_0^T r(t)\cos(2\pi f_c t)dt-\frac{2}{N_0}\sin\psi\int_0^T r(t)\sin(2\pi f_c t)dt\right)$$

$\operatorname{ch}(z_1-z_2)=\operatorname{ch}z_1\operatorname{ch}z_2-\operatorname{sh}z_1\operatorname{sh}z_2$ или
$\operatorname{ch}\left(\frac{2}{N_0}\cos\psi\int_0^T r(t)\cos(2\pi f_c t)dt\right)\cdot\operatorname{ch}\left(\frac{2}{N_0}\sin\psi\int_0^T r(t)\sin(2\pi f_c t)dt\right)-$
$-\operatorname{ch}\left(\frac{2}{N_0}\cos\psi\int_0^T r(t)\cos(2\pi f_c t)dt\right)\cdot\operatorname{sh}\left(\frac{2}{N_0}\sin\psi\int_0^T r(t)\sin(2\pi f_c t)dt\right)$

Аппроксимируем $\operatorname{ch}\left(\frac{2}{N_0}\cos\psi\int_0^T r(t)\cos(2\pi f_c t)\right)dt$ как квадрат того, что под функцией, а $\operatorname{sh}\left(\frac{2}{N_0}\sin\psi\int_0^T r(t)\sin(2\pi f_c t)dt\right)$ как $\left(\frac{2}{N_0}\sin\psi\int_0^T r(t)\sin(2\pi f_c t)dt\right)$ Далее для всех остальных слагаемых, потом возьмем производную.

2) способ.

Аппроксимируем

$$\operatorname{ch}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\varphi_k)dt\right)\approx 1+0.5\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\varphi_k)dt\right)^2$$

И найдем производную. Далее структурную схему.

3) способ. Сначала найдем производную, потом аппроксимируем

$$\operatorname{sh}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\varphi_k)dt\right)\approx\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\varphi_k)dt\right)$$

Получаем $\sum\limits_{m=0}^{k-1}\operatorname{sh}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\right)\cdot\int_0^T r(t)\sin(2\pi f_c t+\psi)dt+$
$\operatorname{sh}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\frac{\pi}{8})dt\right)\int_0^T r(t)\sin(2\pi f_c t+\psi+\frac{\pi}{8}dt+$
$\operatorname{sh}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\frac{\pi}{4})dt\right)\int_0^T r(t)\sin(2\pi f_c t+\psi+\frac{\pi}{4})dt+$
$\operatorname{sh}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\frac{3\pi}{8})dt\right)\int_0^T r(t)\sin(2\pi f_c t+\psi+\frac{3\pi}{8})dt+$
$\operatorname{sh}\left(\frac{2}{N_0}\int_0^T r(t)\sin(2\pi f_c t+\psi)dt\right)\int_0^T r(t)\cos(2\pi f_c t+\psi)dt\approx$
$\sum\limits_{n=0}^{k-1}\sum\limits_{n=0}^{5}\left(\frac{2}{N_0}\int_0^T r(t)\cos(2\pi f_c t+\psi+\varphi_k dt\right)\int_0^T r(t)\sin(2\pi f_c t+\psi+\varphi_k)dt=0;$
Отсюда получается схема устройства.